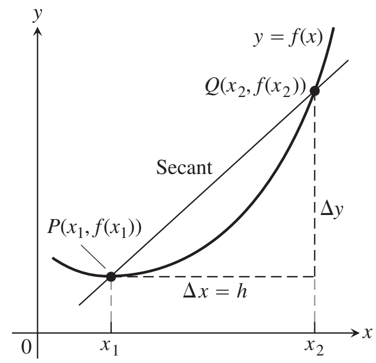
Chapter 2 Limits and Continuity

Average Rate of Change over an Interval

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$



A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

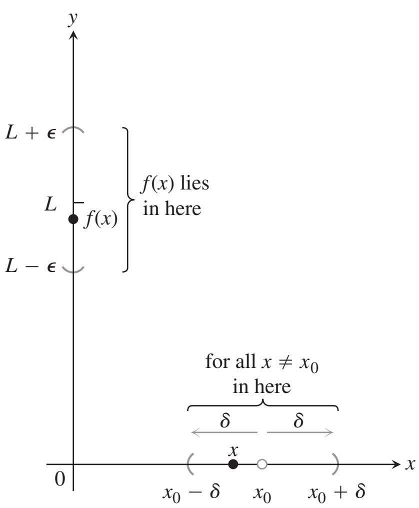
Limits

If $f(x)$ gets arbitrarily close to L (as close to L as we like) for all x sufficiently close to x_0, we say that f approaches the **limit** L as x approaches x_0, and we write

$$\lim_{x \to x_0} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches x_0 is L". Essentially, the definition says that the values of $f(x)$ are close to the number L whenever x is close to x_0 (on either side of x_0).

The formal definition of limit is:



The relation of δ and ϵ in the definition of limit.

Limit of a Function

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

How to Find Algebraically a δ for a Given f, L, x_0, and $\epsilon > 0$

The process of finding a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad |f(x) - L| < \epsilon$$

can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.
2. *Find a value of* $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

Using the formal definition of limit, one can prove the limit laws:

Limit Laws

If L, M, c and k are real numbers and

$$\lim_{x \to c} f(x) = L \qquad \text{and} \qquad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $\lim_{x \to c}(f(x) + g(x)) = L + M$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule*: $\lim_{x \to c}(f(x) - g(x)) = L - M$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule*: $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule*: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule*: If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x \to c}(f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

The Sandwich Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\lim_{x\to c} f(x) = L$.

Note: Analogues of these Laws apply to limits as $x \to \pm\infty$ and one-sided limits.

Examples and Techniques:

a) $\lim\limits_{x\to c} k = k$ (k = constant)

b) $\lim\limits_{x\to c} x^p = c^p$ (where c^p is defined)

c) $\lim\limits_{x\to 1} \dfrac{x^2 + x - 2}{x^2 - x} = \lim\limits_{x\to 1} \dfrac{(x+2)(x-1)}{x(x-1)} = \lim\limits_{x\to 1} (x+2) = 3$ (Canceling a Common Factor)

d) $\lim\limits_{x\to 2} \dfrac{\sqrt{x+7} - 3}{x-2} = \lim\limits_{x\to 2} \dfrac{(\sqrt{x+7}-3)(\sqrt{x+7}+3)}{(x-2)(\sqrt{x+7}+3)}$

$= \lim\limits_{x\to 2} \dfrac{x-2}{(x-2)(\sqrt{x+7}+3)} = \lim\limits_{x\to 2} \dfrac{1}{\sqrt{x+7}+3} = \dfrac{1}{6}$ (Creating and Canceling a Common Factor)

e) $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$ (θ in radians)

f) $\lim\limits_{\theta\to 0} \dfrac{\sin k\theta}{\theta} = k$ (k constant)

g) $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x} = \lim\limits_{x\to 0} = \dfrac{-2\sin^2\left(\frac{x}{2}\right)}{x}$

$= \lim\limits_{x\to 0} (-2)\left(\dfrac{\sin\frac{x}{2}}{x}\right)\sin\left(\dfrac{x}{2}\right)$

$= (-2)\left(\dfrac{1}{2}\right)(0) = 0$ (using **f)** and trigonometric identity: $\cos x = 1 - 2\sin^2\left(\dfrac{x}{2}\right)$ half-angle formula)

One-Sided Limits

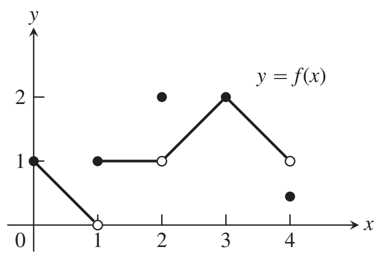
To have a limit L as x approaches c, a function f must be defined on *both sides* of c and its values $f(x)$ must approach L as x approaches c from either side. Because of this, ordinary limits are called **two-sided**.

If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit**. From the left, it is a **left-hand limit**.

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\underset{\text{(2-sided limit)}}{\lim_{x\to c} f(x) = L} \quad \Leftrightarrow \quad \underset{\text{(left-hand limit)}}{\lim_{x\to c^-} f(x) = L} \quad \text{and} \quad \underset{\text{(right-hand limit)}}{\lim_{x\to c^+} f(x) = L.}$$

EXAMPLE



At $x = 0$: $\lim_{x\to 0^+} f(x) = 1$,

$\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.

At $x = 1$: $\lim_{x\to 1^-} f(x) = 0$ even though $f(1) = 1$,

$\lim_{x\to 1^+} f(x) = 1$,

$\lim_{x\to 1} f(x)$ does not exist. The right- and left-hand limits are not equal.

At $x = 2$: $\lim_{x\to 2^-} f(x) = 1$,

$\lim_{x\to 2^+} f(x) = 1$,

$\lim_{x\to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = f(3) = 2$.

At $x = 4$: $\lim_{x\to 4^-} f(x) = 1$ even though $f(4) \neq 1$,

$\lim_{x\to 4^+} f(x)$ and $\lim_{x\to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.

At every other point c in [0, 4], $f(x)$ has limit $f(c)$. ■

Limit as x approaches ∞ or $-\infty$

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x\to\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x\to-\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Horizontal Asymptote

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = b.$$

Infinity, Negative Infinity as Limits

1. We say that ***f*(*x*) approaches infinity as *x* approaches x_0**, and write

$$\lim_{x\to x_0} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad f(x) > B.$$

2. We say that ***f*(*x*) approaches negative infinity as *x* approaches x_0**, and write

$$\lim_{x\to x_0} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad f(x) < -B.$$

Vertical Asymptote

A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x\to a^-} f(x) = \pm\infty.$$

Examples:

h) $\lim_{x\to\pm\infty} k = k \ (k = \text{constant})$

i) For integer $n > 0$,

$$\lim_{x\to\infty} x^n = \infty$$

$$\lim_{x\to-\infty} x^n = \begin{cases} \infty & (n \text{ even}) \\ -\infty & (n \text{ odd}) \end{cases}$$

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$$

$$\lim_{x\to\pm\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m}$$

(provided $a_n, b_m \neq 0$)

j) For integer $n > 0$,

$$\lim_{x\to c^{\pm}} \frac{1}{(x - c)^n} = \begin{cases} +\infty & (n \text{ even}) \\ \pm\infty & (n \text{ odd}) \end{cases}$$

Continuity

Continuous at a Point

Interior point: A function $y = f(x)$ is **continuous at an interior point *c*** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint *a*** or is **continuous at a right endpoint *b*** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively}.$$

Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

A **continuous function** is one that is continuous at every point of its domain.

For example, $y = 1/x$ is not continuous on $[-1, 1]$, but it is continuous over its domain: $(-\infty, 0) \cup (0, \infty)$

The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

Tangents and Derivatives

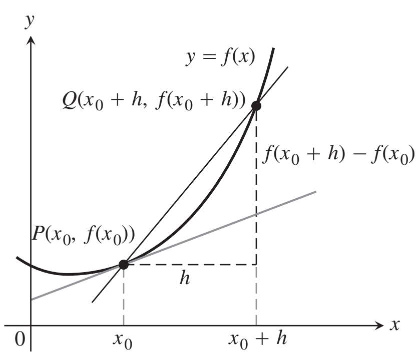


FIGURE 2.67 The slope of the tangent line at P is $\lim_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$.

Slope, Tangent Line

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

Finding the Tangent to the Curve $y = f(x)$ at (x_0, y_0)

1. Calculate $f(x_0)$ and $f(x_0 + h)$.
2. Calculate the slope
$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
3. If the limit exists, find the tangent line as
$$y = y_0 + m(x - x_0).$$

All of these ideas refer to the same thing:

1. The slope of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative of f at $x = x_0$
5. The limit of the difference quotient, $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$