# Chapter 4 Applications of Derivatives

**Absolute Maximum, Absolute Minimum**

Let $f$ be a function with domain $D$. Then $f$ has an **absolute maximum** value on $D$ at a point $c$ if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on $D$ at $c$ if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

**The Extreme Value Theorem**

If $f$ is continuous on a closed interval $[a, b]$, then $f$ attains both an absolute maximum value $M$ and an absolute minimum value $m$ in $[a, b]$. That is, there are numbers $x_1$ and $x_2$ in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other $x$ in $[a, b]$.

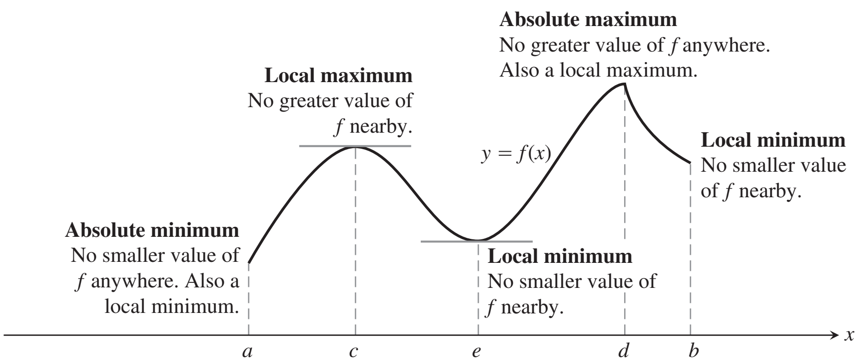
**Local Maximum, Local Minimum**

A function $f$ has a **local maximum** value at an interior point $c$ of its domain if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function $f$ has a **local minimum** value at an interior point $c$ of its domain if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$




**The First Derivative Theorem for Local Extreme Values**

If $f$ has a local maximum or minimum value at an interior point $c$ of its domain, and if $f'$ is defined at $c$, then

$$f'(c) = 0.$$

**Critical Point**

An interior point of the domain of a function $f$ where $f'$ is zero or undefined is a **critical point** of $f$.

**How to Find the Absolute Extrema of a Continuous Function $f$ on a Finite Closed Interval**

1. Evaluate $f$ at all critical points and endpoints.
2. Take the largest and smallest of these values.

**Rolle's Theorem**

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior $(a, b)$. If

$$f(a) = f(b),$$

then there is at least one number $c$ in $(a, b)$ at which

$$f'(c) = 0.$$

**The Mean Value Theorem**

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior $(a, b)$. Then there is at least one point $c$ in $(a, b)$ at which

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

**Increasing, Decreasing Function**

Let $f$ be a function defined on an interval $I$ and let $x_1$ and $x_2$ be any two points in $I$.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then $f$ is said to be **increasing** on $I$.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then $f$ is said to be **decreasing** on $I$.

A function that is increasing or decreasing on $I$ is called **monotonic** on $I$.

**First Derivative Test for Monotonic Functions**

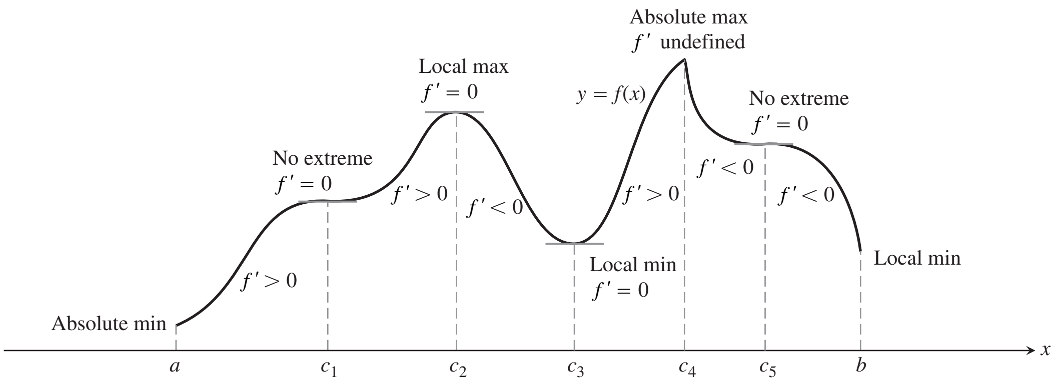
Suppose that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$.

If $f'(x) > 0$ at each point $x \in (a, b)$, then $f$ is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then $f$ is decreasing on $[a, b]$.

**First Derivative Test for Local Extrema**

Suppose that $c$ is a critical point of a continuous function $f$, and that $f$ is differentiable at every point in some interval containing $c$. Moving across $c$ from left to right,

1. if $f'$ changes from negative to positive at $c$, then $f$ has a local minimum at $c$;
2. if $f'$ changes from positive to negative at $c$, then $f$ has a local maximum at $c$;
3. if $f'$ does not change sign at $c$ (that is, $f'$ is positive on both sides of $c$ or negative on both sides), then $f$ has no local extremum at $c$.



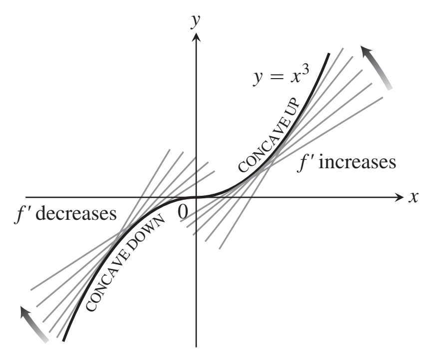

A function's first derivative tells how the graph rises and falls.

**Concave Up, Concave Down**

The graph of a differentiable function $y = f(x)$ is

**(a) concave up** on an open interval $I$ if $f'$ is increasing on $I$

**(b) concave down** on an open interval $I$ if $f'$ is decreasing on $I$.




**The Second Derivative Test for Concavity**

Let $y = f(x)$ be twice-differentiable on an interval $I$.

1. If $f'' > 0$ on $I$, the graph of $f$ over $I$ is concave up.
2. If $f'' < 0$ on $I$, the graph of $f$ over $I$ is concave down.

**Point of Inflection**

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

**Second Derivative Test for Local Extrema**

Suppose $f''$ is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function $f$ may have a local maximum, a local minimum, or neither.

**Strategy for Graphing $y = f(x)$**

1. Identify the domain of $f$ and any symmetries the curve may have.
2. Find $y'$ and $y''$.
3. Find the critical points of $f$, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve.

## Optimization

**Solving Applied Optimization Problems**

1. *Read the problem*. Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture*. Label any part that may be important to the problem.
3. *Introduce variables*. List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. *Write an equation for the unknown quantity*. If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown*. Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

## Indeterminate Forms and L'Hôpital's Rule

**L'Hôpital's Rule**

Suppose that $f(a) = g(a) = 0$, that $f$ and $g$ are differentiable on an open interval $I$ containing $a$, and that $g'(x) \neq 0$ on $I$ if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

**Using L'Hôpital's Rule**

To find

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate $f$ and $g$, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

L'Hôpital's Rule, also applies if $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists. In the notation $x \to a$, $a$ may be either finite or infinite. Moreover $x \to a$ may be replaced by the one-sided limits $x \to a^+$ or $x \to a^-$.

## Indeterminate Forms $\infty/\infty$, $\infty \cdot 0$, $\infty - \infty$

Sometimes when we try to evaluate a limit as $x \to a$ by substituting $x = a$ we get an ambiguous expression like $\infty/\infty$, $\infty \cdot 0$, or $\infty - \infty$, instead of $0/0$. Often we can rewrite these indeterminate forms using algebra to one of the forms $0/0$ or $\infty/\infty$ to which we may apply L'Hôpital's Rule.

## Newton's Method

A numerical method, called *Newton's method* or the *Newton–Raphson method*, is a technique to approximate the solution to an equation $f(x) = 0$.

**Procedure for Newton's Method**

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad \text{if } f'(x_n) \neq 0$$

## Antiderivatives

A function $F$ is an **antiderivative** of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$.

Antiderivative formulas

| | Function | General antiderivative |
|---|---|---|
| **1.** | $x^n$ | $\dfrac{x^{n+1}}{n+1} + C, \quad n \neq -1$, $n$ rational |
| **2.** | $\sin kx$ | $-\dfrac{\cos kx}{k} + C, \quad k$ a constant, $k \neq 0$ |
| **3.** | $\cos kx$ | $\dfrac{\sin kx}{k} + C, \quad k$ a constant, $k \neq 0$ |
| **4.** | $\sec^2 x$ | $\tan x + C$ |
| **5.** | $\csc^2 x$ | $-\cot x + C$ |
| **6.** | $\sec x \tan x$ | $\sec x + C$ |
| **7.** | $\csc x \cot x$ | $-\csc x + C$ |

Antiderivative linearity rules

| | | Function | General antiderivative |
|---|---|---|---|
| **1.** | *Constant Multiple Rule:* | $kf(x)$ | $kF(x) + C, \quad k$ a constant |
| **2.** | *Negative Rule:* | $-f(x)$ | $-F(x) + C,$ |
| **3.** | *Sum or Difference Rule:* | $f(x) \pm g(x)$ | $F(x) \pm G(x) + C$ |

**Indefinite Integral, Integrand**

The set of all antiderivatives of $f$ is the **indefinite integral** of $f$ with respect to $x$, denoted by

$$\int f(x)\, dx.$$

The symbol $\int$ is an **integral sign**. The function $f$ is the **integrand** of the integral, and $x$ is the **variable of integration**.